An elasto-plastic model for non-isothermal analysis of flow and deformation in unsaturated porous media: formulation

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Abstract

A rigorous and unified treatment of the theory of non-isothermal flow and deformation in unsaturated porous media is presented. The governing equations based on the equations of equilibrium, the effective stress concept, Darcy's law, Fourier's law and the conservation equations of mass and energy are derived using a systematic macroscopic approach. The thermo-hydro-mechanical coupling processes taken into account include: thermal expansion, thermal convection by moving fluid, fluid flux due to temperature gradient (Soret effect), phase exchange (vaporisation, condensation), heat exchange between the phases, heat of wetting, and heat due to phase compression. Both elastic and elasto-plastic constitutive equations are developed. All model coefficients are identified in terms of measurable parameters. The governing equations derived are general in nature, embodying all previously presented formulations in the field. For example, when the heat of wetting, and that heat due to phase compression are neglected, and it is assumed that the vapour is at the saturated liquid pressure, with all phases in thermal equilibrium, and that the forced convection is negligible, the theory of heat and mass transfer presented by Thomas and his coworkers is obtained. Also when the pore air volume reduces to zero and the thermal equilibrium is assumed, the thermo-elastic model for fluid saturated media presented by McTigue [J. Geophys. Res. 91 (B9) (1986) 9533] is recovered. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The problem of heat and mass transport in porous media is a subject of great interest in many engineering disciplines. Typical examples include: utilisation of geothermal energy (Jessop, 1998; Kolditz and Clauser, 1998), tertiary enhancement of oil production, remediation of contaminated sites using air sparging (Domenico and Schwartz, 1990; Flynn et al., 1994; Nyer et al., 1996), steam drive/steam flooding (Burger, 1978; Duerksen et al., 1984; Meyer, 1991), in situ combustion (Murdock et al., 1990; Greaves and

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Ibrahim, 1991), and the isolation of hazardous wastes (Chapman and McKinley, 1987). The phenomenon has also bearing on process, mechanical, water resources and agricultural engineering. Over the past two decades, the study of non-isothermal mass flow in porous media and development of strategies for its management and/or containment in natural formations and engineered materials has been a key area of research in modern environmental geomechanics. Many attempts have been made to develop predictive capabilities associated with topics such as the storage of nuclear wastes, remediation of contaminated sites, movement of pollutant plumes, etc.

Several constitutive models have been proposed. Philip and de Vries (1957) presented a model for coupled heat and moisture transfer in rigid porous media under the combined gradients of temperature and moisture. de Vries (1958) extended this theory to include moisture and latent heat storage in the vapour phase, and the advection of sensible heat by water. Modified versions of Philip-de Vries model were also proposed by Sophocleous (1978), Milly (1982), Bear et al. (1991), Thomas and King (1991) and Thomas and Sansom (1995), using matric suction rather than volumetric moisture content as the primary variable. The laboratory and field validation of Philip and de Vries theory has been reported by Rose (1968), Ewen and Thomas (1989) and Thomas and Li (1997), among others. Reasonable agreement has been found between the theoretical analyses and the laboratory/field results. Geraminegad and Saxena (1986) presented a de-coupled flow–deformation model, including the effect of matrix deformation on moisture, heat and gas flow through the porous media. They defined the mechanical behaviour of the soil in terms of the ''stress state surface'' and the ''independent stress state variables'', in which the total stress tensor in excess of air pressure (net stress) and suction are considered independent (Matyas and Radhakrishna, 1968; Fredlund and Morgenstern, 1977). A coupled version of this formulation was later presented by Thomas and He (1995, 1997). They introduced matrix displacement vector as a primary variable, and improved the coupling effects between the temperature and deformation. They also improved the energy balance equation by including moisture and latent heat storage in the vapour phase, in addition to the advection of heat by water previously accounted for by de Vries (1958). Similar formulations have also been given by Gawin et al. (1995), Gawin and Schrefler (1995), Gens et al. (1998) and Zhou et al. (1998, 1999). However, in Gawin et al. (1995) and Gawin and Schrefler (1995) the constitutive laws of the solid phase were introduced through the concept of effective stress. Nevertheless, they used the degree of saturation, as the effective stress parameter, which is not fully supported by the experimental evidence. They also retained the degree of saturation as the main coupling element between the air and water flow fields, rendering the governing differential equations strongly non-linear.

In general, a major difficulty in the formulations discussed above is that they either completely ignore the matrix deformation (e.g. Philip and de Vries, 1957; de Vries, 1958; Milly, 1982; Bear et al., 1991; Thomas and King, 1991; Thomas and Sansom, 1995; Nassar and Horton, 1997) or use the theory of elasticity (Gawin et al., 1995; Gawin and Schrefler, 1995) in conjunction with the ''state surfaces'' approach (Thomas and He, 1995; Gens et al., 1998; Zhou et al., 1998) to account for the strongly non-linear deformation behaviour of the soil matrix. It is well established that vastly different volumetric responses can be obtained in an unsaturated soil, subject to identical increases in matric suction and net stress, but applied at different sequences. Such stress path dependency cannot be modelled using the theory of elasticity and/or the state surfaces approach. An appropriate plasticity model must be invoked, in order to take into account the variation of the yield surface with temperature and suction. Furthermore, in these formulations, the effect of temperature on the state surfaces is not well defined, despite its importance in the deformation response of the soil (e.g. see Hueckel and Baldi, 1990; Sultan, 1997).

Another difficulty in the formulations above is that they often neglect the advective flow of vapour and do not consider the transfer of heat between the phases, which can be significant particularly in applications involving fractured rock formations or processes which require injection or extraction of steam/heated air into or out of natural formations. A detailed account of the constraints associated with the local thermal equilibrium assumption may be found in Quintard and Whitaker (1995). Also assumed in these formula-

tions is the spontaneous thermo-dynamic equilibrium between the soil liquid and the water vapour, rendering the vapour pressure a dependent variable of suction and temperature, thus allowing the flux of vapour and liquid water to be presented using a single equation (Thomas and He, 1995; Gens et al., 1998). Although convenient from a modelling point of view, this assumption is restrictive and may only be applicable to processes which are thermally induced and diffusion dominated. In fact, most of the models presented in the literature appear to have been developed for a specific application, namely a heat generating canister located in a low permeability formation.

The major objective of this paper is to present a more complete treatment of the theory of heat and mass transport through deformable unsaturated porous media. The work is an extension of the theoretical developments of Loret and Khalili (2000a) for fully coupled isothermal flow through variably saturated deformable porous media to include thermal coupling effects. The governing equations are developed using a systematic macroscopic approach, satisfying the equations of balance of mass, momentum and energy. Three phases (solid, s, liquid, l, and gas, g) and four constituents (solid, s, water, w, vapour, v, and dry air, da) are identified. Each phase is viewed as an independent continuum, endowed with its own kinematics, mass, momentum, temperature and energy. Thermal non-equilibrium between the phases is assumed throughout. The solid constituent and the fluid are a priori compressible, but either one can be made incompressible, if necessary. Matrix displacement vector, pore gas pressure, pore water pressure, pore vapour pressure, solid temperature, liquid temperature and gas temperature are introduced as the nine primary variables in a three-dimensional boundary value problem. The system is closed using the static equilibrium or balance of momentum of the entire mixture (three equations), the balance of momentum of the fluid phase or Darcy's law (two equations), the balance of energy of each of the three phases (three equations) and the balance of mass of vapour (one equation). The physical processes taken into account for the purposes of constitutive modelling include: mechanical deformation, thermal expansion, advection, diffusion (Fick's law), heat conduction (Fourier's law), liquid flux due to temperature gradient (Soret effect), phase exchange (vapourisation, condensation), heat convection (by moving liquid and moving gas), heat exchange between the phases, heat of wetting, heat due to phase compression and heat transfer due to phase exchange (latent heat). The potential effect of damage due to suction and desiccation induced cracking is ignored. Both elastic and elasto-plastic constitutive equations are discussed. Special attention is placed on identifying model parameters in terms of measurable coefficients. Emphasis is also placed on the interaction between the phases both in the elastic and plastic regimes.

2. Conceptual model

Conceptually, the formulation presented consists of four separate, yet overlapping models. These include: (1) deformation model, (2) flow model, (3) vapour transport model, and (4) heat transfer model.

The deformation model is based on a thermo-elastic–plastic framework, satisfying the equations of equilibrium. The yield surface is defined in the effective stress space but it is affected by the plastic volumetric strain, matric suction, and temperature. The constitutive equations in both elastic and plastic regimes are written in terms of effective stress rather than the independent stress state variables as used in previous models (Alonso et al., 1990). The advantage of using the effective stress is that the elastic deformation of the system can completely be expressed in terms of a single "effective" stress variable rather than two or three independent stress variables. This significantly simplifies the deformation model, and hence reduces the model parameters (Loret and Khalili, 2001). Furthermore, with an effective stress based model, the transition from saturation to unsaturation or vice versa can be readily taken into account using a single set of constitutive equations (Loret and Khalili, 2000a). For a discussion on the validity of the effective stress principle in unsaturated soils the reader is referred to Khalili et al. (2001).

The flow model is based on two interacting continua: one representing the liquid flow and the other the gas flow. Thus, at every point in space, two pressures are introduced: one denoting the average liquid pressure – at the neighbourhood of the given point – and the other the average gas pressure. The interaction between the two flow fields is established through the phase exchange term, describing the transfer of water from liquid phase to gas phase in a vapour form, and vice versa, and through the soil water characteristic curve describing the dependency of the volume fraction occupied by each phase (or the degree of saturation) on suction at a given temperature. The rate of phase transfer at any point is assumed to be a function of the difference between the saturated vapour pressure and the partial vapour pressure within the gas phase. The coupling between the flow and the deformation fields is established through the effective stress parameters, α_l and α_g, which relate the change in liquid and gas pressures to the change in the matrix deformation vector.

The vapour transport model is based on two processes: diffusion due to the vapour density gradient, and advection due to the bulk flow of the gas. The vapour is treated as an ideal gas and as a constituent within the gas phase. The link between the vapour transport model and the liquid flow model is established through the mass exchange term and the psychrometric law. The coupling with the deformation model is achieved through the gas phase and the effective stress parameter α_g.

The heat transfer model is based on the assumption that energy is transferred through all three phases and that the phases are not at thermal equilibrium. At any point, three temperatures are introduced; one for each of the phases of solid, liquid and gas. Vapour and dry air are assumed to be at the same temperature. The energy transfer processes considered include: conduction, advection or convection (i.e. by the moving phases), heat transfer due to phase exchange, heat exchange between the phases, heat of wetting, and heat generated due to phase compression. The thermal interaction between the phases is established through the phase exchange terms. The coupling between the heat transfer model and the flow and deformation models is established through the heat of wetting and the heat of compression. For the flow model, an additional coupling is also established through the heat transfer due to phase exchange (latent heat).

3. Deformation model

To describe the solid phase deformation in an unsaturated thermo-elastic medium, consider a representative elementary volume of the medium, subject to external (total) stresses, σ_{ij}. The linear momentum balance equations for this elemental volume, neglecting the inertial effects, are written as,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \quad i = 1, \ldots, N \tag{1}$$

where F_i is ith component of the body force per unit volume in the orthogonal spatial coordinates, x_i, with $i = 1, \ldots, N$, and N is the space dimension. Extending Terzaghi's effective stress to unsaturated soils, and considering the effects of pore gas and pore liquid pressures separately, the total stress in an incremental form may be expressed as,

$$d\sigma_{ij} = d\sigma'_{ij} - \alpha_l \, dp_l \delta_{ij} - \alpha_g \, dp_g \delta_{ij} \tag{2}$$

where σ'_{ij} is the effective stress, p_l is the pore liquid pressure, p_g is the pore gas pressure (positive for compressive pressure), α_l and α_g are the incremental effective stress parameters, and δ_{ij} is the Kronecker's delta. Following and extending Nur and Byerlee (1971) analysis for isotropic saturated soils, the effective stress parameters α_l and α_g can be defined as, Khalili and Khabbaz (1995) and Loret and Khalili (2000a),

$$\alpha_l = \frac{c_m}{c} - \frac{c_s}{c} \quad \text{and} \quad \alpha_g = 1 - \frac{c_m}{c} \tag{3}$$

where c_s is the compressibility of solid grains, c is the drained compressibility of the soil structure, and c_m represents the tangent compressibility of the soil structure with respect to a change in matric suction $s = p_g - p_l$. For incompressible solid grains, Eq. (3) yields $\alpha_l + \alpha_g = 1$. Notice that at full saturation, $c_m = c$, thus, $\alpha_l = 1$ and $\alpha_g = 0$, and Eq. (2) reduces to Terzaghi's effective stress for saturated soils.

For a thermo-elastic medium the stress–temperature–strain relationship is written as,

$$d\sigma'_{ij} = D_{ijkl}\, d\varepsilon_{kl} - \gamma\, dT_s \delta_{ij} \tag{4}$$

where D_{ijkl} is the drained stiffness of the soil skeleton, with symmetries in the first two and the last two indices, ε_{ij} is the soil skeleton strain, T_s is the temperature of the solid constituent, and $\gamma = -1/3(\partial\sigma'_{mm}/\partial T_s)|_{\varepsilon_{kk}}$, which is related to the volumetric thermal expansion of the medium, c_T, through $\gamma = c_T/c$. Implicit in Eq. (4) is the assertion that the thermal expansion of the solid skeleton is controlled by the temperature of the solid constituent only. This is in contrast with the work of Pecker and Deresiewicz (1973), in which the thermal expansion of the solid skeleton for saturated porous media is related to a combination of solid and pore fluid temperatures. To explore this aspect further, consider an elemental volume of a porous medium subject to constituent temperatures T_s, T_l and T_g, with T_l and T_g denoting the temperature of the liquid and gas constituents occupying the pore space, respectively (Fig. 1). Let the solid constituent be completely sealed from the thermal effects within the pore space. It is evident that in such a system the "drained" thermal expansion/contraction of the solid skeleton will be independent of the temperature changes within the pore space. Extending this model to real soils, it is concluded that the drained thermal volumetric change in the solid skeleton is independent of the change in the pore gas and pore liquid temperatures, until energy exchange occurs between the constituents, giving rise to a change in the temperature of the solid. This latter effect, which is a matter of conservation energy, is included in the energy balance equations of solid, liquid and gas.

Now, using the relationship between the strain and displacement,

$$d\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\, du_i}{\partial x_j} + \frac{\partial\, du_j}{\partial x_i}\right) \tag{5}$$

where u_i denotes the displacement components of the solid skeleton, Eqs. (1)–(5) can be combined to yield,

$$\frac{\partial}{\partial x_j}\left[D_{ijkl}\frac{\partial\, du_k}{\partial x_l} - (\alpha_l\, dp_l + \alpha_g\, dp_g + \gamma\, dT_s)\delta_{ij}\right] + dF_i = 0 \tag{6}$$
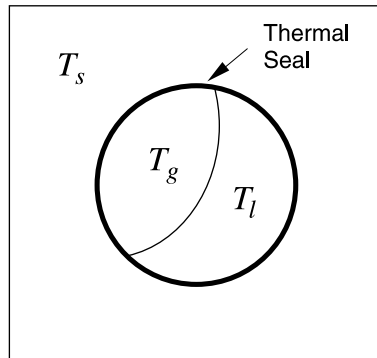

Fig. 1. Schematic representation of a three phase thermo-elastic porous element.

The set of differential equations (6), for $i = 1, \ldots, N$, governs the deformation phenomenon in an unsaturated thermo-elastic medium.

4. Flow model

4.1. Liquid phase

The most common method of modelling liquid flow in unsaturated media is to combine the equation of linear momentum balance for the liquid phase with the mass balance equation of the fluid. Neglecting the inertial and viscous effects, the equation of linear momentum balance for the liquid phase can be written as,

$$v_{li}^r = -\frac{k_{lij}}{\mu_l}\left(\frac{\partial p_l}{\partial x_j} + \rho_l g_j\right) - k_{lTij}\frac{\partial T_l}{\partial x_j}, \quad i = 1, \ldots, N \tag{7}$$

where k_{lij} is the intrinsic permeability of the liquid phase and k_{lTij} is the coefficient of thermal coupling for liquid flux (thermo-osmosis), analogous to the Soret effect in molecular diffusion, in which mass flux is caused due to both concentration gradient and temperature gradient. μ_l and ρ_l are the dynamic viscosity and the density of liquid respectively, and g_i is the ith component of gravitational acceleration. v_{li}^r is the relative velocity of liquid defined as,

$$v_{li}^r = n_l(v_{li} - v_{si}) \tag{8}$$

where n_l is the volumetric liquid content, v_{li} is the absolute liquid velocity and v_{si} is the solid skeleton velocity,

$$v_{li} = \frac{\partial u_{li}}{\partial t} \quad \text{and} \quad v_{si} = \frac{\partial u_i}{\partial t} \tag{9}$$

where u_{li} and u_i are the components of displacements of fluid and solid, respectively.

Satisfying the conservation of fluid mass yields,

$$-\frac{\partial}{\partial x_i}(\rho_l n_l v_{li}) = \frac{\partial}{\partial t}(n_l \rho_l) + \rho_l j_{lg} \tag{10}$$

where j_{lg} denotes the rate of moisture transfer between the liquid phase and the gas phase (negative for condensation and positive for vapourisation), which can be described using Dalton's equation as (e.g. see Marshall and Holmes, 1988),

$$j_{lg} = \omega(p_{sv} - p_v) \tag{11}$$

in which $\omega > 0$ is the liquid phase transfer coefficient, p_v is the vapour pressure, and p_{sv} is the saturated vapour pressure which can be obtained using the psychrometric law (Kelvin's equation), e.g. Stallman (1964),

$$p_{sv} = p_{svo} \exp\left(\frac{-sM_l}{\rho_l R T_g}\right) \tag{12}$$

where M_l is the molecular mass of liquid ($= 18$ g/mol for water), R ($= 8.134$ J/mol/K) is the universal gas constant, s is the suction, and p_{svo} is the saturated vapour pressure in contact with the liquid over a planar surface. p_{svo} is a function of temperature only. It can be obtained using the empirical vapour pressure data published in the literature (e.g. Maidment, 1992), or calculated based on Clausius–Clapeyron equation, e.g. Kestin (1966),

$$p_{\text{svo}} = \varsigma \exp\left(\frac{-LM_l}{RT_g}\right) \tag{13}$$

in which L is the latent heat of vapourisation, and $\varsigma = p_{\text{svo}}^* \exp(LM_l/RT^*)$, with T^* and p_{svo}^* being the temperature and the saturated vapour pressure in contact with water over a planar surface, respectively, at a reference configuration.

Now, substituting Eq. (8) into Eq. (10),

$$-\frac{\partial}{\partial x_i}(\rho_l v_{li}^r) - \frac{\partial}{\partial x_i}(\rho_l n_l v_{si}) = \frac{\partial}{\partial t}(\rho_l n_l) + \rho_l j_{lg} \tag{14}$$

and introducing the Lagrangian total derivative with respect to moving solid, $d_s(\cdot)/dt = \partial(\cdot)/\partial t + v_{si}\partial(\cdot)/\partial x_i$, and moving liquid, $d_l(\cdot)/dt = \partial(\cdot)/\partial t + v_{li}\partial(\cdot)/\partial x_i$, Eq. (14) can be rearranged to,

$$-\rho_l \frac{\partial}{\partial x_i}(v_{li}^r) = n_l \frac{d_l \rho_l}{dt} + \rho_l \frac{d_s n_l}{dt} + \rho_l n_l \frac{\partial v_{si}}{\partial x_i} + \rho_l j_{lg} \tag{15}$$

If the liquid is barotropic, $\rho_l = \rho_l(p_l, T_l)$, then,

$$\frac{d_l \rho_l}{dt} = \rho_l c_l \frac{d_l p_l}{dt} - \rho_l c_{lT} \frac{d_l T_l}{dt} \tag{16}$$

in which

$$c_l = \frac{1}{\rho_l} \left.\frac{\partial \rho_l}{\partial p_l}\right|_{T_l} \quad \text{and} \quad c_{lT} = -\frac{1}{\rho_l} \left.\frac{\partial \rho_l}{\partial T_l}\right|_{p_l}$$

are the coefficients of compressibility and thermal expansion for liquid, respectively. Using the definition of volumetric liquid content, $n_l = V_l/V$,

$$\frac{d_s n_l}{dt} = \frac{1}{V}\left(\frac{d_s V_l}{dt} - n_l \frac{d_s V}{dt}\right) \tag{17}$$

Eqs. (7), (15), (16) and (17) can be combined to yield,

$$\frac{\partial}{\partial x_i}\left[\frac{k_{lij}}{\mu_l}\left(\frac{\partial p_l}{\partial x_j} + \rho_l g_j\right) + k_{lTij}\frac{\partial T_l}{\partial x_j}\right] = n_l c_l \frac{d_l p_l}{dt} + \frac{1}{V}\frac{d_s V_l}{dt} - n_l c_{lT}\frac{d_l T_l}{dt} + j_{lg} \tag{18}$$

In Eq. (18), $d_s V_l/V$ represents the change in the pore liquid space over the current volume of the porous medium, which for isothermal conditions can be calculated from (Khalili and Khabbaz, 1995; Khalili and Valliappan, 1996),

$$\frac{d_s V_l}{V} = (c_m - c_s) d_s\left(\frac{\sigma_{ii}}{3} + p_g\right) - c_m' d_s(p_g - p_l) - n_l c_s d_s p_g \tag{19}$$

where $c_m' = -(d_s V_l/V)/ds|_{\sigma_{ii}, T, p_g}$ represents the change in the volumetric pore liquid content with suction, which can be obtained using the slope of the soil water characteristic curve (Loret and Khalili, 2000a). Extending Eq. (19) to include thermal effects, we have,

$$\frac{d_s V_l}{V} = (c_m - c_s) d_s\left(\frac{\sigma_{ii}}{3} + p_g\right) - c_m' d_s(p_g - p_l) - n_l c_s d_s p_g + \xi_{lgT} d_s T + n_l c_T d_s T_s \tag{20a}$$

in which ξ_{lgT} denotes the change in the pore liquid space due to a change in the curvature of the liquid–gas interface, arising from the temperature dependency of the interfacial surface energy. T is the average temperature acting on the liquid–gas interface, approximated as $T = n_s T_s + n_g T_g + n_l T_l$, in which n_s is the volumetric solid content, n_g is the volumetric gas (dry air and vapour) content, and n_l is the volumetric liquid content subject to the constraint $n_s + n_g + n_l = 1$. T_s is the temperature of the solid constituent, T_g is

the gas temperature, T_l is the liquid temperature. The last term on the right-hand side of Eq. (20a) represents the change in the volumetric pore liquid due to the overall change in the volume of the solid skeleton with temperature.

From Eqs. (2)–(4),

$$d_s\left(\frac{\sigma_{ii}}{3} + p_g\right) = \frac{1}{c}(d_s\varepsilon_{jj} - c_T\, d_s T_s) + \alpha_l\, d_s(p_g - p_l) + \frac{c_m}{c}\, d_s p_g \tag{20b}$$

Eq. (20a) can be rearranged to,

$$\frac{d_s V_l}{V} = \alpha_l\, d_s\varepsilon_{ii} + (c'_m - \alpha_l^2 c)\, d_s p_l - (c'_m - \alpha_l^2 c - \alpha_l c_s + n_l c_s)\, d_s p_g - (\alpha_l - n_l)c_T\, d_s T_s + \xi_{lgT}\, d_s T \tag{20c}$$

Now, substituting Eq. (20b) into Eq. (18), noting $(1/V)d_s V/dt = \partial v_{si}/\partial x_i$, and introducing the approximation $v_{si}\partial(\cdot)/\partial x_i \ll \partial(\cdot)/\partial t$ or $d_s(\cdot)/dt \approx \partial(\cdot)/\partial t$, the differential equation governing non-isothermal liquid flow through an unsaturated porous medium becomes,

$$\frac{\partial}{\partial x_i}\left[\frac{k_{lij}}{\mu_l}\left(\frac{\partial p_l}{\partial x_j} + \rho_l g_j\right) + k_{lTij}\frac{\partial T_l}{\partial x_j}\right] = (n_l c_l + c'_m - \alpha_l^2 c)\frac{\partial p_l}{\partial t} - (c'_m - \alpha_l^2 c - \alpha_l c_s + n_l c_s)\frac{\partial p_g}{\partial t} + \alpha_l\frac{\partial^2 u_i}{\partial t\,\partial x_i}$$
$$- n_l c_{lT}\frac{\partial T_l}{\partial t} - n_l c_{lT} v_{li}\frac{\partial T_l}{\partial x_i} + n_l c_l v_{li}\frac{\partial p_l}{\partial x_i} - (\alpha_l - n_l)c_T\frac{\partial T_s}{\partial t} + \xi_{lgT}\frac{\partial T}{\partial t} + j_{lg} \tag{21}$$

4.2. Gas phase

The mixture of air and water vapour is treated as a homogenous gas (i.e. the dry air and the vapour occupy the same volume fraction n_g, and are completely miscible). The mass averaged advective velocity of air–vapour mixture with respect to the moving solids, $v^r_{gi} = n_g(v_{gi} - v_{si})$, is due to both pressure and temperature gradients, that is,

$$v^r_{gi} = -\frac{k_{gij}}{\mu_g}\frac{\partial p_g}{\partial x_j} - k_{gTij}\frac{\partial T_g}{\partial x_j} \tag{22a}$$

where k_{gij} and μ_g are the components of the permeability tensor and the dynamic viscosity of air–vapour mixture, respectively, and k_{gTij} are the components of the thermal coupling tensor for gas flux.

In Eq. (22a), the mass averaged velocity of the gas, v_{gi}, is defined as,

$$v_{gi} = \frac{1}{\rho_g}(\rho_{da} v_{dai} + \rho_v v_{vi}) \tag{22b}$$

where v_{dai} is the absolute velocity of the dry air, v_{vi} is the absolute velocity of the vapour, and

$$\rho_g = \rho_{da} + \rho_v \tag{22c}$$

ρ_g is the density of the gas, ρ_{da} is the density of the dry air, and ρ_v is the density of the vapour. The volumetric gas content, n_g, is defined as,

$$n_g \equiv \frac{V_g}{V} = n - n_l = \frac{\rho_l}{\rho_l - \rho_v}[(1 - S_r)n - HnS_r] \tag{23}$$

where H is the coefficient of solubility of gas in liquid defined by Henry's law, S_r is the degree of saturation defined as the volume of water (i.e. volume of liquid plus condensed vapour) over the volume of voids (i.e. V_w/V_v), and n is the porosity $V_v/V = n_g + n_l$.

The diffusive velocities of the dry air and the vapour with respect to the mass averaged velocity of gas are defined as,

$$\widehat{v}_{\mathrm{da}i} = v_{\mathrm{da}i} - v_{\mathrm{g}i} = \frac{1}{n_{\mathrm{g}}}(v_{\mathrm{da}i}^{\mathrm{r}} - v_{\mathrm{g}i}^{\mathrm{r}}) \tag{24a}$$

$$\widehat{v}_{\mathrm{v}i} = v_{\mathrm{v}i} - v_{\mathrm{g}i} = \frac{1}{n_{\mathrm{g}}}(v_{\mathrm{v}i}^{\mathrm{r}} - v_{\mathrm{g}i}^{\mathrm{r}}) \tag{24b}$$

and they satisfy the constraint,

$$\rho_{\mathrm{da}}\widehat{v}_{\mathrm{da}i} + \rho_{\mathrm{v}}\widehat{v}_{\mathrm{v}i} = 0 \tag{24c}$$

In Eqs. (24a)–(24c), $v_{\mathrm{da}i}^{\mathrm{r}}$ and $v_{\mathrm{v}i}^{\mathrm{r}}$ are the relative dry air and vapour velocities with respect to the moving solid.

To satisfy the conservation of mass for the gas phase, we first write the conservation of mass for the dry air and vapour as,

Dry air

$$-\frac{\partial}{\partial x_i}(\rho_{\mathrm{da}}n_{\mathrm{g}}v_{\mathrm{da}i}) = \frac{\partial}{\partial t}(n_{\mathrm{g}}\rho_{\mathrm{da}}) \tag{25}$$

Vapour

$$-\frac{\partial}{\partial x_i}(\rho_{\mathrm{v}}n_{\mathrm{g}}v_{\mathrm{v}i}) = \frac{\partial}{\partial t}(n_{\mathrm{g}}\rho_{\mathrm{v}}) - \rho_{\mathrm{l}}j_{\mathrm{lg}} \tag{26}$$

The conservation equation of mass for the gas phase is simply obtained by summing the balance Eqs. (25) and (26), yielding,

$$-\frac{\partial}{\partial x_i}(\rho_{\mathrm{g}}n_{\mathrm{g}}v_{\mathrm{g}i}) = \frac{\partial}{\partial t}(n_{\mathrm{g}}\rho_{\mathrm{g}}) - \rho_{\mathrm{l}}j_{\mathrm{lg}} \tag{27}$$

Now, following the same procedure as for the liquid phase, Eq. (27) can be expanded and re-arranged to,

$$-\rho_{\mathrm{g}}\frac{\partial}{\partial x_i}(v_{\mathrm{g}i}^{\mathrm{r}}) = n_{\mathrm{g}}\frac{\mathrm{d}_{\mathrm{g}}\rho_{\mathrm{g}}}{\mathrm{d}t} + \rho_{\mathrm{g}}\frac{\mathrm{d}_{\mathrm{s}}n_{\mathrm{g}}}{\mathrm{d}t} + \rho_{\mathrm{g}}n_{\mathrm{g}}\frac{\partial v_{\mathrm{s}i}}{\partial x_i} - \rho_{\mathrm{l}}j_{\mathrm{lg}} \tag{28}$$

The dry air–vapour mixture is considered as an ideal gas,

$$\rho_{\mathrm{g}} = \frac{P_{\mathrm{g}}M_{\mathrm{g}}}{RT_{\mathrm{g}}} \tag{29}$$

where M_{g} is the molecular mass of air–vapour mixture, with $\rho_{\mathrm{g}}/M_{\mathrm{g}} = (\rho_{\mathrm{v}}/M_{\mathrm{l}}) + (\rho_{\mathrm{a}}/M_{\mathrm{a}})$, $P_{\mathrm{g}} = p_{\mathrm{g}} + p_{\mathrm{atm}}$ is the absolute gas pressure. The derivative of ρ_{g} with respect to the moving gas in Eq. (28) can be written as,

$$\frac{\mathrm{d}_{\mathrm{g}}\rho_{\mathrm{g}}}{\mathrm{d}t} = \rho_{\mathrm{g}}c_{\mathrm{g}}\frac{\mathrm{d}_{\mathrm{g}}p_{\mathrm{g}}}{\mathrm{d}t} - \rho_{\mathrm{g}}c_{\mathrm{gT}}\frac{\mathrm{d}_{\mathrm{g}}T_{\mathrm{g}}}{\mathrm{d}t} \tag{30}$$

in which $c_{\mathrm{g}} = 1/P_{\mathrm{g}}$, and $c_{\mathrm{gT}} = 1/T_{\mathrm{g}}$. Substituting Eqs. (22a) and (30) into Eq. (28) gives,

$$\frac{\partial}{\partial x_i}\left(\frac{k_{\mathrm{g}ij}}{\mu_{\mathrm{g}}}\frac{\partial p_{\mathrm{g}}}{\partial x_j} + k_{\mathrm{gT}ij}\frac{\partial T_{\mathrm{g}}}{\partial x_j}\right) = n_{\mathrm{g}}c_{\mathrm{g}}\frac{\partial p_{\mathrm{g}}}{\partial t} + \frac{1}{V}\frac{\mathrm{d}_{\mathrm{s}}V_{\mathrm{g}}}{\mathrm{d}t} - n_{\mathrm{g}}c_{\mathrm{gT}}\frac{\partial T_{\mathrm{g}}}{\partial t} - n_{\mathrm{g}}c_{\mathrm{gT}}v_{\mathrm{g}i}\frac{\partial T_{\mathrm{g}}}{\partial x_i} + n_{\mathrm{g}}c_{\mathrm{g}}v_{\mathrm{g}i}\frac{\partial p_{\mathrm{g}}}{\partial x_i} - \frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{g}}}j_{\mathrm{lg}} \tag{31}$$

Introducing the constitutive relationship (Khalili and Khabbaz, 1995),

$$\frac{\mathrm{d}_{\mathrm{s}}V_{\mathrm{g}}}{V} = \alpha_{\mathrm{g}}\varepsilon_{ii} + [c_{\mathrm{m}}' - \alpha_{\mathrm{l}}^2 c - (\alpha_{\mathrm{l}} - \alpha_{\mathrm{g}} + n_{\mathrm{g}} - n_{\mathrm{l}})c_{\mathrm{s}}]\mathrm{d}_{\mathrm{s}}p_{\mathrm{g}}$$
$$- [c_{\mathrm{m}}' - \alpha_{\mathrm{l}}^2 c - (\alpha_{\mathrm{l}} - n_{\mathrm{l}})c_{\mathrm{s}}]\mathrm{d}_{\mathrm{s}}p_{\mathrm{l}} - (\alpha_{\mathrm{g}} - n_{\mathrm{g}})c_{\mathrm{T}}\mathrm{d}_{\mathrm{s}}T_{\mathrm{s}} - \xi_{\mathrm{lgT}}\mathrm{d}_{\mathrm{s}}T \tag{32}$$

and assuming $v_{\mathrm{s}i}(\partial(\cdot)/\partial x_i) \ll \partial(\cdot)/\partial t$, Eq. (31) can be written as,

$$\frac{\partial}{\partial x_i}\left(\frac{k_{gij}}{\mu_g}\frac{\partial p_g}{\partial x_j}+k_{gTij}\frac{\partial T_g}{\partial x_j}\right)=[n_gc_g+c'_m-\alpha_1^2c-(\alpha_1-\alpha_g+n_g-n_1)c_s]\frac{\partial p_g}{\partial t}-[c'_m-\alpha_1^2c-(\alpha_1-n_1)c_s]\frac{\partial p_1}{\partial t}$$

$$+\alpha_g\frac{\partial^2 u_i}{\partial t\,\partial x_i}-n_gc_{gT}\frac{\partial T_g}{\partial t}-n_gc_{gT}v_{gi}\frac{\partial T_g}{\partial x_i}+n_gc_gv_{gi}\frac{\partial p_g}{\partial x_i}-(\alpha_g-n_g)c_T\frac{\partial T_s}{\partial t}$$

$$-\xi_{lgT}\frac{\partial T}{\partial t}-\frac{\rho_1}{\rho_g}j_{lg} \tag{33}$$

5. Vapour transport

The vapour transport within the gas phase is described as the sum of two processes: (1) diffusion due to vapour gradient, (2) advection due to the movement of the gas. Following the Fickian concept of mass transport by molecular diffusion, the diffusive vapour mass flux, q^d_{vi}, due to the vapour gradient is written as,

$$q^d_{vi}=\rho_vn_g\widehat{v}_{vi}=\rho_v(v^r_{vi}-v^r_{gi})=-n_gD_{vij}\frac{\partial\rho_v}{\partial x_j} \tag{34}$$

in which D_{vij} is the molecular diffusivity of vapour in the pore gas. The advective vapour mass flux with respect to the moving solid is,

$$q^a_{vi}=\rho_vv^r_{gi} \tag{35}$$

Rewriting the conservation of vapour mass (26) in order to highlight the advective and diffusive transfer terms, we obtain using Eq. (24b),

$$-\frac{\partial}{\partial x_i}(q^d_{vi}+q^a_{vi})-\frac{\partial}{\partial x_i}(n_g\rho_vv_{si})=\frac{\partial}{\partial t}(n_g\rho_v)-\rho_1j_{lg} \tag{36}$$

Notice that the second term on the left-hand side of Eq. (36) describes the conservation of vapour mass with respect to a fixed reference system rather than the moving solid. For the purpose of combining with the constitutive equations, Eq. (36) is rearranged to,

$$-\rho_v\frac{\partial}{\partial x_i}\left[\frac{1}{\rho_v}(q^d_{vi}+q^a_{vi})\right]=n_g\frac{d_v\rho_v}{dt}+\rho_v\frac{d_sn_g}{dt}+\rho_vn_g\frac{\partial v_{si}}{\partial x_i}-\rho_1j_{lg} \tag{37}$$

Approximating $\partial V_g/\partial t\approx d_sV_g/dt$, i.e. $(1/V)\partial V_g/\partial t\approx d_sn_g/dt+n_g(\partial v_{si}/\partial x_i)$, considering vapour as a perfect gas $\rho_v=P_vM_1/RT_g$, and using Eqs. (32), (34) and (35), Eq. (37) can be further expanded in terms of the primary variables,

$$\frac{\partial}{\partial x_i}\left(n_gc_vD_{vij}\frac{\partial p_v}{\partial x_j}\right)+\frac{\partial}{\partial x_i}\left[(k_{gTij}-n_gc_{vT}D_{vij})\frac{\partial T_g}{\partial x_j}\right]+\frac{\partial}{\partial x_i}\left(\frac{k_{gij}}{\mu_g}\frac{\partial p_g}{\partial x_j}\right)$$

$$=n_gc_v\frac{\partial p_v}{\partial t}+[c'_m-\alpha_1^2c-(\alpha_1-\alpha_g+n_g-n_1)c_s]\frac{\partial p_g}{\partial t}-[c'_m-\alpha_1^2c-(\alpha_1-n_1)c_s]\frac{\partial p_1}{\partial t}+\alpha_g\frac{\partial^2 u_i}{\partial t\,\partial x_i}$$

$$-n_gc_{vT}\frac{\partial T_g}{\partial t}-n_gc_{vT}v_{vi}\frac{\partial T_g}{\partial x_i}+n_gc_vv_{vi}\frac{\partial p_v}{\partial x_i}-(\alpha_g-n_g)c_T\frac{\partial T_s}{\partial t}-\xi_{lgT}\frac{\partial T}{\partial t}-\frac{\rho_1}{\rho_v}j_{lg} \tag{38}$$

in which $P_v=p_v+p_{atm}$ is the absolute vapour pressure, $c_v=1/P_v$, and $c_{vT}=c_{gT}=1/T_g$.

6. Heat transport

Energy balance equations for small changes in temperature are given by Pecker and Deresiewicz (1973), Bowen and Chen (1975), Bear and Corapcioglu (1981), McTigue (1986), Bear et al. (1991) and Gens et al. (1998). Assuming the solid, liquid and gas phase temperatures are not in local thermal equilibrium, and neglecting the increase in the internal energy due to shearing, the energy balance equation of the solid phase is written as,

$$
\frac{\partial}{\partial x_i}\left(n_s \lambda_{sij}\frac{\partial T_s}{\partial x_j}\right) = n_s \rho_s C_s\left(\frac{\partial T_s}{\partial t}+v_{si}\frac{\partial T_s}{\partial x_i}\right) - T_s\left[(\alpha_l - n_l)c_T - n_s\xi_{lgT}\right]\frac{\partial p_l}{\partial t}
$$
$$
- T_s\left[(\alpha_g - n_g)c_T + n_s\xi_{lgT}\right]\frac{\partial p_g}{\partial t} + T_s\gamma\frac{\partial^2 u_i}{\partial t\,\partial x_i} + \kappa_{sl}(T_s - T_l) + \kappa_{sg}(T_s - T_g) \tag{39}
$$

in which λ_{sij} is thermal conductivity of solid phase, C_s is the heat capacity of the solid, κ_{sl} is the coefficient of heat exchange between the solid and liquid phases, and κ_{sg} is the coefficient of heat exchange between the solid and gas phases. Terms considered in Eq. (39) are the flux of thermal energy due to conduction and convection, the change in the energy due to a change in matric suction (i.e. heat of wetting plus the work of pore air and pore water pressures on the two pore boundaries), change in the internal energy of the solid phase due to volume change of solid skeleton, and the exchange of energy between the solid phase and the liquid phase and between the solid phase and the gas phase. Notice that the terms on the right-hand side of Eq. (39) are positive when they correspond to loss of energy for the solid phase.

The energy balance equation for the liquid phase is in turn written as,

$$
\frac{\partial}{\partial x_i}\left(n_l \lambda_{lij}\frac{\partial T_l}{\partial x_j}\right) = C_l\left(\frac{\partial n_l\rho_l T_l}{\partial t}+\frac{\partial v_{li}n_l\rho_l T_l}{\partial x_i}\right) + n_l T_l\beta_l\frac{d_l\varepsilon_l}{dt}\bigg|_{T_l} - n_l T_l\xi_{lgT}\left(\frac{\partial p_g}{\partial t}-\frac{\partial p_l}{\partial t}\right) + L\rho_l\langle j_{lg}\rangle
$$
$$
+ \kappa_{ls}(T_l - T_s) + \kappa_{lg}(T_l - T_g) \tag{40}
$$

with

$$
\frac{d_l\varepsilon_l}{dt}\bigg|_{T_l} = -\frac{1}{\rho_l}\frac{d_l\rho_l}{dt}\bigg|_{T_l} = -c_l\frac{d_l p_l}{dt} \tag{41a}
$$

in which λ_{lij} and C_l are the thermal conductivity and the heat capacity of the liquid phase, respectively. $\kappa_{ls} = \kappa_{sl}$ and κ_{lg} is the coefficient of heat exchange between the liquid phase and the gas phase, L is the latent heat of vapourisation, and $\beta_l = \partial p_l/\partial T_l|_{\varepsilon_l} = c_{lT}/c_l$. The third term on the right-hand side represents the change in the internal energy due to change in the volume of liquid, the fourth term represents the change in the energy due to a change in matric suction, and the fifth term represents the energy loss due to phase change. Notice that this latter term will only be present during vapourisation, as indicated by the symbol $\langle\cdot\rangle$; namely $\langle j_{lg}\rangle = j_{lg}$ if $j_{lg} \geqslant 0$ and $\langle j_{lg}\rangle = 0$ if $j_{lg} < 0$.

Now, considering

$$
\frac{\partial n_l\rho_l T_l}{\partial t}+\frac{\partial v_{li}n_l\rho_l T_l}{\partial x_i} = T_l\left(\frac{\partial n_l\rho_l}{\partial t}+\frac{\partial v_{li}n_l\rho_l}{\partial x_i}\right) + n_l\rho_l\left(\frac{\partial T_l}{\partial t}+v_{li}\frac{\partial T_l}{\partial x_i}\right) \tag{41b}
$$

together with Eq. (41a), Eq. (40) can be written as,

$$
\frac{\partial}{\partial x_i}\left(n_l \lambda_{lij}\frac{\partial T_l}{\partial x_j}\right) = n_l C_l\rho_l\frac{\partial T_l}{\partial t} - n_l\left(c_{lT}-\xi_{lgT}\right)T_l\frac{\partial p_l}{\partial t} - n_l\xi_{lgT}T_l\frac{\partial p_g}{\partial t} + L\rho_l\langle j_{lg}\rangle - C_l\rho_l j_{lg}T_l + \kappa_{ls}(T_l - T_s)
$$
$$
+ \kappa_{lg}(T_l - T_g) + n_l C_l\rho_l v_{li}\frac{\partial T_l}{\partial x_i} \tag{42}
$$

Similarly, the energy balance equation for the gas phase can be written as,

$$\frac{\partial}{\partial x_i}\left(n_g\lambda_{gij}\frac{\partial T_g}{\partial x_j}\right) = n_g C_g\rho_g\frac{\partial T_g}{\partial t} + n_g\xi_{lgT}T_g\frac{\partial p_l}{\partial t} - n_g(c_{gT} + \xi_{lgT})T_g\frac{\partial p_g}{\partial t} - L\rho_l\langle -j_{lg}\rangle + C_g\rho_l j_{lg}T_g$$

$$+ \kappa_{gs}(T_g - T_s) + \kappa_{gl}(T_g - T_l) + n_g C_g\rho_g v_{gi}\frac{\partial T_g}{\partial x_i} \tag{43}$$

in which λ_{gij} and C_g are the thermal conductivity and the heat capacity of the gas phase, respectively. Again notice that in Eq. (42) energy generated due to phase exchange occurs only during condensation, and $\kappa_{gs} = \kappa_{sg}$, $\kappa_{gl} = \kappa_{lg}$. Finally, we note that in Eqs. (42) and (43) the term $C_\alpha T_\alpha$ could be replaced by the enthalpy h_α of the phase α.

7. Summary of governing equations – elastic behaviour

In summary, the differential equations describing the coupled thermo-hydro-elastic process in unsaturated media may be written as,

Deformation – balance of momentum of the mixture (N equations)

$$\frac{\partial}{\partial x_j}\left[D_{ijkl}\frac{\partial du_k}{\partial x_l} - (\alpha_l dp_l + \alpha_g dp_g + \gamma dT_s)\delta_{ij}\right] + dF_i = 0 \tag{44a}$$

Liquid flow – balance of mass of fluid (one equation)

$$\frac{\partial}{\partial x_i}\frac{k_{lij}}{\mu_l}\left(\frac{\partial p_l}{\partial x_j} + \rho_l g_j\right) + \frac{\partial}{\partial x_i}\left(k_{lTij}\frac{\partial T_l}{\partial x_j}\right) = a_{ll}\frac{\partial p_l}{\partial t} - a_{lg}\frac{\partial p_g}{\partial t} + \alpha_l\frac{\partial^2 u_i}{\partial t\partial x_i} - a_{lT_s}\frac{\partial T_s}{\partial t} - a_{lT_l}\frac{\partial T_l}{\partial t} - a_{lT_g}\frac{\partial T_g}{\partial t}$$

$$- n_l c_{lT}v_{li}\frac{\partial T_l}{\partial x_i} + n_l c_l v_{li}\frac{\partial p_l}{\partial x_i} + j_{lg} \tag{44b}$$

Gas flow – balance of mass of gas phase (one equation)

$$\frac{\partial}{\partial x_i}\left(\frac{k_{gij}}{\mu_g}\frac{\partial p_g}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(k_{gTij}\frac{\partial T_g}{\partial x_j}\right) = a_{gg}\frac{\partial p_g}{\partial t} - a_{gl}\frac{\partial p_l}{\partial t} + \alpha_g\frac{\partial^2 u_i}{\partial t\partial x_i} - a_{gT_s}\frac{\partial T_s}{\partial t} - a_{gT_l}\frac{\partial T_l}{\partial t} - a_{gT_g}\frac{\partial T_g}{\partial t}$$

$$- n_g c_{gT}v_{gi}\frac{\partial T_g}{\partial x_i} + n_g c_g v_{gi}\frac{\partial p_g}{\partial x_i} - \frac{\rho_l}{\rho_g}j_{lg} \tag{44c}$$

Vapour transport – balance of mass of vapour (one equation)

$$\frac{\partial}{\partial x_i}\left(n_g c_v D_{vij}\frac{\partial p_v}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left[(k_{gTij} - n_g c_{vT}D_{vij})\frac{\partial T_g}{\partial x_j}\right] + \frac{\partial}{\partial x_i}\left(\frac{k_{gij}}{\mu_g}\frac{\partial p_g}{\partial x_j}\right)$$

$$= a_{vv}\frac{\partial p_v}{\partial t} + a_{vg}\frac{\partial p_g}{\partial t} - a_{vl}\frac{\partial p_l}{\partial t} + \alpha_g\frac{\partial^2 u_i}{\partial t\partial x_i} - a_{vT_s}\frac{\partial T_s}{\partial t} - a_{vT_l}\frac{\partial T_l}{\partial t} - a_{vT_g}\frac{\partial T_g}{\partial t} - a_{vT_g}v_{vi}\frac{\partial T_g}{\partial x_i}$$

$$+ a_{vv}v_{vi}\frac{\partial p_g}{\partial x_i} - \frac{\rho_l}{\rho_g}j_{lg} \tag{44d}$$

Heat transport – balance of energy (three equations)
   Solid phase

$$\frac{1}{T_s}\frac{\partial}{\partial x_i}\left(n_s\lambda_{sij}\frac{\partial T_s}{\partial x_j}\right) = a_{T_sT_s}\frac{\partial T_s}{\partial t} - a_{T_s1}\frac{\partial p_1}{\partial t} - a_{T_sg}\frac{\partial p_g}{\partial t} + a_{T_s\varepsilon}\frac{\partial^2 u_i}{\partial t\,\partial x_i} + \frac{\kappa_{sl}}{T_s}(T_s - T_1) + \frac{\kappa_{sg}}{T_s}(T_s - T_g) + a_{T_sT_s}v_{si}\frac{\partial T_s}{\partial x_i}$$
(44e)

Fluid phase

$$\frac{1}{T_1}\frac{\partial}{\partial x_i}\left(n_1\lambda_{1ij}\frac{\partial T_1}{\partial x_j}\right) = a_{T_1T_1}\frac{\partial T_1}{\partial t} - a_{T_11}\frac{\partial p_1}{\partial t} - a_{T_1g}\frac{\partial p_g}{\partial t} + \frac{L\rho_1}{T_1}\langle j_{lg}\rangle - C_1\rho_1 j_{lg} + \frac{\kappa_{ls}}{T_1}(T_1 - T_s) + \frac{\kappa_{lg}}{T_1}(T_1 - T_g)$$
$$+ a_{T_1T_1}v_{li}\frac{\partial T_1}{\partial x_i}$$
(44f)

Gas phase

$$\frac{1}{T_g}\frac{\partial}{\partial x_i}\left(n_g\lambda_{gij}\frac{\partial T_g}{\partial x_j}\right) = a_{T_gT_g}\frac{\partial T_g}{\partial t} - a_{T_g1}\frac{\partial p_1}{\partial t} - a_{T_gg}\frac{\partial p_g}{\partial t} - \frac{L\rho_1}{T_g}\langle -j_{lg}\rangle + C_g\rho_1 j_{lg} + \frac{\kappa_{gs}}{T_g}(T_g - T_s)$$
$$+ \frac{\kappa_{gl}}{T_g}(T_g - T_1) + a_{T_gT_g}v_{gi}\frac{\partial T_g}{\partial x_i}$$
(44g)

subject to the constitutive equations,

$$\frac{d\sigma_{ii}}{3} = c^{-1}\,d\varepsilon_{kk} - \alpha_1\,dp_1 - \alpha_g\,dp_g - a_{\varepsilon T_s}\,dT_s$$
(45a)

$$\frac{dm_1}{\rho_{lo}} = \alpha_1\,d\varepsilon_{ii} + a_{11}\,dp_1 - a_{1g}\,dp_g - a_{1T_s}\,dT_s - a_{1T_1}\,dT_1 - a_{1T_g}\,dT_g$$
(45b)

$$\frac{dm_g}{\rho_{go}} = \alpha_g\,d\varepsilon_{ii} - a_{gl}\,dp_1 + a_{gg}\,dp_g - a_{gT_s}\,dT_s - a_{gT_1}\,dT_1 - a_{gT_g}\,dT_g$$
(45c)

$$d\eta_s = a_{T_s\varepsilon}\,d\varepsilon_{ii} - a_{T_s1}\,dp_1 - a_{T_sg}\,dp_g + a_{T_sT_s}\,dT_s$$
(45d)

$$d\eta_1 = -a_{T_11}\,dp_1 - a_{T_1g}\,dp_g + a_{T_1T_1}\,dT_1$$
(45e)

$$d\eta_g = -a_{T_g1}\,dp_1 - a_{T_gg}\,dp_g + a_{T_gT_g}\,dT_g$$
(45f)

$$j_{lg} = \omega(p_{sv} - p_v)$$
(45g)

with

$$\alpha_l = \frac{c_m}{c} - \frac{c_s}{c}, \quad \alpha_g = 1 - \frac{c_m}{c}$$

$$a_{ll} = n_l c_l + c'_m - \alpha_l^2 c, \quad a_{gg} = n_g c_g + c'_m - \alpha_l^2 c - (\alpha_l - \alpha_g + n_g - n_l) c_s, \quad a_{vv} = n_g c_v$$

$$a_{T_s T_s} = \frac{n_s \rho_s C_s}{T_s}, \quad a_{T_l T_l} = \frac{n_l \rho_l C_l}{T_l}, \quad a_{T_g T_g} = \frac{n_g C_g \rho_g}{T_g}$$

$$a_{lg} = a_{gl} = a_{vl} = c'_m - \alpha_l^2 c - (\alpha_l - n_l) c_s, \quad a_{vg} = c'_m - \alpha_l^2 c - (\alpha_l - \alpha_g + n_g - n_l) c_s \tag{46}$$

$$a_{\varepsilon T_s} = a_{T_s \varepsilon} = \gamma, \quad a_{l T_s} = a_{T_s l} = (\alpha_l - n_l) c_T - n_s \xi_{lgT}$$

$$a_{l T_l} = a_{T_l l} = n_l (c_{lT} - \xi_{lgT}), \quad a_{l T_g} = a_{T_g l} = -n_g \xi_{lgT}$$

$$a_{g T_s} = a_{T_s g} = a_{v T_s} = (\alpha_g - n_g) c_T + n_s \xi_{lgT}, \quad a_{g T_l} = a_{T_l g} = a_{v T_l} = n_l \xi_{lgT}$$

$$a_{g T_g} = a_{T_g g} = n_g (c_{gT} + \xi_{lgT}), \quad a_{v T_g} = n_g (c_{vT} + \xi_{lgT})$$

in which m_l and m_g are the mass liquid and the mass gas contents per reference volume of the porous medium, respectively. η_s, η_l and η_g are the entropies per unit volume of the solid, liquid and gas respectively. ρ_{lo} and ρ_{go} are the densities of the liquid and gas at the reference configuration, respectively. The constitutive equations in Eqs. (45a)–(45g) enjoy the major symmetry due to the reversibility assumption of the physical processes taken into account. This result can also be achieved, through thermo-dynamic arguments, within the general framework of mixture theory (Loret and Khalili, 2000b).

Noting that in many practical problems the compressibility of solid grains, c_s, is negligible, then the coefficients in Eq. (3) simplify to,

$$\alpha_l = \frac{c_m}{c}, \quad \alpha_g = 1 - \alpha_l \tag{47a}$$

and the constitutive coefficients describing the hydro-mechanical response reduce to,

$$a_{ll} = n_l c_l + a_{gl}, \quad a_{gg} = n_g c_g + a_{gl}$$
$$a_{lg} = a_{gl} = a_{vl} = a_{vg} = c'_m - \alpha_l^2 c \tag{47b}$$

The governing equations presented above are general in nature and embody most of the previously presented formulations in the field. For instance, when the heat of wetting, and heat due to phase compression are neglected, and it is assumed that the vapour is at the saturated liquid pressure, with all the phases in thermal equilibrium, and that the vapour transport is due to diffusion only, the theory of heat and mass transfer presented by Thomas and his coworkers is obtained. Also when the pore air volume reduces to zero (i.e. $S_r = 1$, $c = c_m = c'_m$ and $\xi_{lgT} = 0$) and the thermal equilibrium is assumed, the thermo-elastic model for fluid saturated media presented by McTigue (1986) is recovered.

8. Parameter identification

The proposed model involves the following parameters:

(a) D_{ijkl} (unit: Pa) is the drained stiffness of the soil skeleton, with symmetries in the first two and the last two indices. For an isotropic elastic material, D_{ijkl} is completely defined in terms of the drained modulus of elasticity, E, and the drained Poisson's ratio, v.
(b) α_l and α_g are the non-dimensional tangent effective stress parameters, relating pore liquid and pore gas pressures to matrix deformation.
(c) a_{ll} and a_{gg} (unit: Pa^{-1}) are the apparent compressibilities of liquid and gas, respectively.
(d) $a_{lg} = a_{gl}$ (unit: Pa^{-1}) is the coupling term relating the pore gas and pore liquid volumetric deformations due to a change in suction.

(e) $a_{T_sT_s}$, $a_{T_lT_l}$ and $a_{T_gT_g}$ (unit: Pa/K^2) are the apparent heat capacities of the solid phase, the liquid phase and the gas phase respectively.

(f) $a_{\varepsilon T_s} = a_{T_s\varepsilon}$ (unit: Pa/K) is the thermal expansion coefficient providing coupling between the deformation and the heat transport models.

(g) $a_{lT_s} = a_{T_sl}$, $a_{lT_l} = a_{T_ll}$, $a_{lT_g} = a_{T_gl}$ (unit: 1/K) provide coupling between the liquid flow model and the energy balance equations of solid, liquid and gas, respectively, arising from the temperature dependency of the liquid–gas interface and the heat of wetting.

(h) $a_{gT_s} = a_{T_sg}$, $a_{gT_l} = a_{T_lg}$, $a_{gT_g} = a_{T_gg}$ (unit: 1/K) provide coupling between the gas flow and the energy balance equations of solid, liquid and gas, respectively.

(i) a_{vl} and a_{vg} (unit: Pa^{-1}) describe the change in the pore vapour volume due to a change in liquid and gas pressures, respectively.

(j) a_{vv} (unit: Pa^{-1}) is the apparent compressibility of the vapour.

(k) a_{vT_s}, a_{vT_l}, a_{vT_g} (unit: 1/K) provide coupling between the vapour transport model and the energy balance equations.

(l) $\kappa_{sl} = \kappa_{ls}$, $\kappa_{lg} = \kappa_{gl}$ and $\kappa_{gs} = \kappa_{sg}$ (unit: Pa/K/s) are the coefficients of heat exchange between solid–liquid, liquid–gas, and gas–solid phases, respectively.

(m) ω (unit: Pa^{-1}/s) is a proportionality coefficient controlling the rate of water transfer from liquid to vapour and vice versa.

(n) D_{vij} (unit: m^2/s) is the molecular diffusivity of vapour in the pore gas.

(o) λ_{sij}, λ_{lij} and λ_{gij} (unit: N/K/s) are the thermal conductivity of the solid, liquid and gas phases.

(p) k_{lij} and k_{gij} (unit: m^2) are the unsaturated permeability tensors of the soil with respect to liquid and gas, respectively.

(q) k_{lTij} and k_{gTij} (unit: m^2/K/s) are tensors of thermal coupling for liquid and gas flux, respectively.

(r) ξ_{lgT} (unit: 1/K) denotes the change in the pore liquid space due to a change in the curvature of the liquid–gas interface.

(s) μ_l and μ_g (unit: Pa s) are the dynamic viscosities of liquid and gas, respectively.

The coefficients α_l, α_g, a_{ll}, a_{gg}, $a_{lg} = a_{gl}$, $a_{T_sT_s}$, $a_{T_lT_l}$, $a_{T_gT_g}$, $a_{\varepsilon T_s} = a_{T_s\varepsilon}$, $a_{lT_s} = a_{T_sl}$, $a_{lT_l} = a_{T_ll}$, $a_{lT_g} = a_{T_gl}$, $a_{gT_s} = a_{T_sg}$, $a_{gT_l} = a_{T_lg}$, $a_{gT_g} = a_{T_gg}$, a_{vl}, a_{vg}, a_{vv}, a_{vT_s}, a_{vT_l} and a_{vT_g} are all related to the basic measurable parameters n, S_r, c, c_s, c_l, c_m, c'_m, C_s, C_l, C_g, c_T, c_{lT}, $c_{gT} = c_{vT}$, ρ_s, ρ_l, ρ_g and ξ_{lgT} through relations given in Eq. (46). n and S_r can be measured using a number of direct and indirect methods available in the literature, e.g. Bear (1972), with S_r also being linked to matric suction through several relationships based on the soil water characteristic curve (see e.g. van Genuchten, 1980). The compressibility coefficients of water and solid, c_l and c_s, as well as the densities of solid, ρ_s, liquid ρ_l, and gas, ρ_g, are readily available from the literature, with c_s often assumed to be zero. Similarly, the thermal conductivity coefficients λ_{sij}, λ_{lij} and λ_{gij}, the heat capacity coefficients C_s, C_l, C_g and the thermal expansion coefficients c_T, c_{lT}, c_{gT}, c_{vT}, can be obtained from the literature, see e.g. de Vries (1963) and Bear (1972). c and c_m are the compressibility coefficients of the soil structure with respect to total stress and matric suction, respectively. Depending on strain conditions, the compressibility coefficient, c, for an isotropic elastic medium, is related to E and v, according to the following relationships,

for one-dimensional strain condition, $\varepsilon_{11} = c\sigma_{11}$,

$$c = \frac{(1 - 2v)(1 + v)}{(1 - v)E} \tag{48a}$$

for two-dimensional strain condition, $\varepsilon_{11} + \varepsilon_{22} = c(\sigma_{11} + \sigma_{22})/2$,

$$c = \frac{2(1 - 2v)(1 + v)}{E} \tag{48b}$$

for three-dimensional strain condition, $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = c(\sigma_{11} + \sigma_{22} + \sigma_{33})/3$,

$$c = \frac{3(1-2v)}{E} \tag{48c}$$

Alternatively, the compressibility coefficients, c and c_{m} can be determined directly by subjecting a representative elementary volume of the soil to an isothermal change in total stress or matric suction and monitoring the volumetric change. In this case c and c_{m} are defined as (Eqs. (45a) and (46)),

$$c = \frac{\mathrm{d}V/V}{\mathrm{d}(\sigma_{ii}/3 + p_{\mathrm{g}})} \quad \text{for } \mathrm{d}s = 0, \ \mathrm{d}p_{\mathrm{l}} = 0 \tag{49}$$

$$c_{\mathrm{m}} = -\frac{\mathrm{d}V/V}{\mathrm{d}s} \quad \text{for } \mathrm{d}\left(\frac{\sigma_{ii}}{3} + p_{\mathrm{g}}\right) = 0, \ \mathrm{d}p_{\mathrm{l}} = 0 \tag{50}$$

The effective stress parameters α_{l} and α_{g}, and therefore the compressibility coefficient c_{m}, can also be determined conveniently from the "total" effective stress parameter, χ, proposed by Khalili and Khabbaz (1998), and the effective stress equation (Bishop, 1959),

$$\begin{aligned} \sigma'_{ij} &= \sigma_{ij} + (\chi_{\mathrm{l}}p_{\mathrm{l}} + \chi_{\mathrm{g}}p_{\mathrm{g}})\delta_{ij} \\ \chi_{\mathrm{l}} &= \chi \ \text{and} \ \chi_{\mathrm{g}} = 1 - \chi - \frac{c_{\mathrm{s}}}{c} \end{aligned} \tag{51}$$

with

$$\chi = \begin{cases} [s_{\mathrm{e}}(T)/s]^{\Omega} & \text{for } s \geqslant s_{\mathrm{e}}(T) \\ 1 & \text{for } s \leqslant s_{\mathrm{e}}(T) \end{cases} \tag{52a}$$

in which $s_{\mathrm{e}}(T)$ is the bubbling pressure or the air entry value at the temperature, T, and $\Omega = 0.55$. The air entry value at temperature T can be expressed in terms of the air entry value at the reference temperature T_{o} as,

$$s_{\mathrm{e}}(T) = \frac{\sigma_{\mathrm{T}}}{\sigma_{T_{\mathrm{o}}}} s_{\mathrm{e}}(T_{\mathrm{o}}) \tag{52b}$$

where σ_{T} is the surface tension energy at temperature T which is approximated as (Edlefsen and Anderson, 1943),

$$\sigma = 0.1171 - 0.0001516T \ (\text{unit } \mathrm{J}\,\mathrm{m}^{-2}) \tag{52c}$$

Writing the incremental form of Eq. (51) and comparing to Eq. (2), we have,

$$\alpha_{\mathrm{l}} = \frac{\partial(\chi s)}{\partial s} = \begin{cases} [(1-\gamma)\chi] & \text{for } s \geqslant s_{\mathrm{e}}(T) \\ 1 & \text{for } s < s_{\mathrm{e}}(T) \end{cases} \tag{53a}$$

$$\alpha_{\mathrm{g}} = 1 - \alpha_{\mathrm{l}} - \frac{c_{\mathrm{s}}}{c} \tag{53b}$$

The compressibility coefficient of the liquid phase with respect to an isothermal change in matric suction, c'_{m}, can in turn be determined as (Eq. (19)),

$$c'_m = -\frac{dV_l/V}{ds} \quad \text{for } d\left(\frac{\sigma_{ii}}{3} + p_g\right) = 0, \ dp_g = 0 \tag{54}$$

c'_m can also be determined from the soil water characteristic curve using (Loret and Khalili, 2000a),

$$c'_m = -G_s(1-n)\frac{dw}{ds} + \frac{n_l^2}{(1-n)}c_s \tag{55}$$

in which G_s is the specific gravity, w is the conventional water content defined as the mass of water over the mass of solid.

The coefficients of heat exchange, $\kappa_{sl} = \kappa_{ls}$, $\kappa_{lg} = \kappa_{gl}$ and $\kappa_{gs} = \kappa_{sg}$, can be determined experimentally by systematically subjecting each of the phases (solid, gas, liquid) to a change in the temperature and monitoring the response of the other phases within the system with time, much in line with the experimental analysis of Pecker and Deresiewicz (1973) for fluid saturated porous media. By conducting the tests at various degrees of saturation a relationship can be established between the heat exchange coefficients and the saturation state of the system. For each degree of saturation, three independent tests will be required to characterise each of the coefficients.

k_{lij} and k_{gij} can be determined directly in the laboratory or indirectly by the methods mentioned in the literature, such as Yoshimi and Osterberg (1963), Brooks and Corey (1966) and Bear (1972). In addition, the non-linear behaviour of permeability with void ratio can be taken into account, if required. Using the concept of the relative permeabilities of soil to flow of water and air, k_{lij} and k_{gij} are often written as,

$$k_{lij} = k_{ij} \times k_{rl} \tag{56}$$

$$k_{gij} = k_{ij} \times k_{rg} \tag{57}$$

where k_{ij} is the intrinsic permeability tensor of soil and, k_{rl} and k_{rg} are the relative permeabilities of water and air respectively. The following semi-empirical relationships between k_{rl} and k_{rg} and water saturation are proposed by Brooks and Corey (1966),

$$k_{rl} = S_e^{(2+3\lambda_p)/\lambda_p} \tag{58}$$

$$k_{rg} = (1-S_e)^2(1-S_e^{(2+\lambda_p)/\lambda_p}) \tag{59}$$

where S_e is the effective degree of saturation, defined as,

$$S_e = \frac{S_r - S_{ru}}{1 - S_{ru}} \ \text{ or } \ S_e = \left(\frac{s_e}{s}\right)^{\lambda_p} \tag{60}$$

in which S_{ru} is the residual degree of saturation and λ_p is the pore size distribution index. Both of these parameters are identified from the soil-water characteristic curve presented in terms of S_r.

The thermal coupling coefficients k_{lTij}, k_{gTij}, k_{Tlij} and k_{Tgij} can also be determined in the laboratory. Typical values may be found in Mitchell (1993).

The coefficient ξ_{lgT} can be obtained by conducting a series of suction-controlled and stress-controlled heating/cooling tests on a representative sample of the soil – using a modified pressure plate equipment – and monitoring the mass liquid content of the soil. For a given temperature, ξ_{lgT} can then be calculated using the constitutive relations (45a) and (45b) as,

$$\xi_{lgT} = \frac{dm_l}{\rho_{lo}dT} - n_l(c_T - c_{lT}) \tag{61}$$

9. Elastic–plastic constitutive equations

9.1. Basic assumptions

To describe the plastic properties of the system, we adopt the assumption that the yield function is defined in the effective stress space, but its size, which in the Cam-Clay framework is measured by the pre-consolidation stress, is affected by suction as observed in the experiments by Alonso et al. (1990) and by temperature as in Hueckel and Baldi (1990) and Sultan (1997), among others. More specifically, the yield function is defined as,

$$f = f(\sigma'_{ij}, s, T_s, X) \tag{62}$$

in which X is a set of hardening parameters. The yield function f is assumed to be smooth at the point of interest. The plastic behaviour emanates from the solid skeleton. However, all three phases (solid, liquid, and gas) will develop irreversible strains as in the case of saturated poro-plastic media (Rice, 1975). The plastic flow is defined through a generalised normality rule and the existence of a plastic potential such that $g = g(\sigma_{ij}, p_l, p_g, T_s) = g(\sigma'_{ij}, s, T_s)$, and

$$d\varepsilon^{pl}_{ij} = d\Lambda \frac{\partial g}{\partial \sigma_{ij}}, \qquad d\eta^{pl}_s = d\Lambda \frac{\partial g}{\partial T_s}, \qquad \frac{dm^{pl}_J}{\rho_{Jo}} = d\Lambda \frac{\partial g}{\partial p_J} \quad J = l, g \tag{63}$$

$d\Lambda \geqslant 0$ is the plastic multiplier and the superscript pl denotes the plastic response. The behaviour is called associative, and the thermo-elastic–plastic stiffness of the underlying drained solid will display the major symmetry property when the directions of the normals to the yield surface and the plastic potential in the space $(\sigma_{ij}, p_l, p_g, T_s)$ are identical.

Using the definition of the effective stress, the following straightforward connections are established:

$$\frac{\partial g}{\partial \sigma_{ij}} = \frac{\partial g}{\partial \sigma'_{ij}}, \qquad \frac{\partial g}{\partial p_l} = \alpha_l \frac{\partial g}{\partial \sigma'_{ii}} - \frac{\partial g}{\partial s}, \qquad \frac{\partial g}{\partial p_g} = \alpha_g \frac{\partial g}{\partial \sigma'_{ii}} + \frac{\partial g}{\partial s} \tag{64}$$

Similar connections hold for the yield function f. Thus,

$$\frac{dm^{pl}_l}{\rho_{lo}} = \alpha_l \varepsilon^{pl}_{ii} - d\Lambda \frac{\partial g}{\partial s}, \qquad \frac{dm^{pl}_g}{\rho_{go}} = \alpha_g \varepsilon^{pl}_{ii} + d\Lambda \frac{\partial g}{\partial s} \tag{65}$$

Further simplifying assumptions can be made by imposing the restriction that the underlying behaviour of the solid skeleton follows the lines of the critical state theory, Schofield and Wroth (1968). Then, the volumetric plastic strains vanish at critical state. However, the fluid mass content will continue to vary if $\partial g / \partial s$ is different from zero, which is physically inadmissible as the fluid volume content must follow the same trend. Therefore, in the absence of the experimental evidence to the contrary, we assume,

$$\frac{\partial g}{\partial s} = 0 \tag{66}$$

or

$$\frac{dm^{pl}_l}{\rho_{lo}} = \alpha_l \varepsilon^{pl}_{ii}, \quad \frac{dm^{pl}_g}{\rho_{go}} = \alpha_g \varepsilon^{pl}_{ii} \tag{67}$$

A direct consequence of Eq. (66) is that the behaviour is non-associative, and that the elastic–plastic constitutive equations will not be symmetric.

Finally, we assume that the increments of strain, fluid mass contents and entropy of solid are additively decomposed into elastic and a plastic part,

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{el} + d\varepsilon_{ij}^{pl}, \qquad dm_J = dm_J^{el} + dm_J^{pl} \quad J = l, g$$

$$d\eta_s = d\eta_s^{el} + d\eta_s^{pl}$$

(68)

9.2. Incremental elastic–plastic constitutive equations

Now, to obtain the incremental elastic–plastic constitutive equations of the system, the consistency equation is first written in the usual way,

$$df = \frac{\partial f}{\partial \sigma'_{ij}} d\sigma'_{ij} + \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial T_s} dT_s + \frac{\partial f}{\partial X} dX = 0$$

(69)

Substituting Eqs. (4), (63) and the first term of Eq. (64) into Eq. (69) yields the plastic multiplier $d\Lambda$,

$$d\Lambda = \frac{1}{H} \left[\frac{\partial f}{\partial \sigma'_{ij}} D_{ijkl} d\varepsilon_{kl} + \frac{\partial f}{\partial s} ds + \left(\frac{\partial f}{\partial T_s} - \frac{\partial f}{\partial \sigma'_{ij}} D_{ijkk} \gamma \right) dT_s \right]$$

(70)

in which the modulus $H = H(s, T_s, X)$, which depends a priori on suction and the solid temperature, is defined as,

$$H \equiv h + \frac{\partial f}{\partial \sigma'_{ij}} D_{ijkl} \frac{\partial g}{\partial \sigma'_{kl}} > 0$$

(71)

It is noted that H is formally identical to its isothermal counterpart in the underlying drained solid $H = H|_{s=0,T_s}$. The hardening modulus $h = h(s, T_s, X)$ is defined as,

$$h \equiv -\frac{\partial f}{\partial X} \frac{dX}{d\Lambda}$$

(72)

which describes the evolution of the size of the yield surface and is a priori a function of suction and solid temperature. Substituting the plastic multiplier $d\Lambda$, Eq. (70), into the elastic equations in Eqs. (45a)–(45g) and making use of the relations (65) and (68), we obtain the incremental elastic–plastic equations,

$$d\sigma_{ij} = D_{ijkl}^{ep} d\varepsilon_{kl} - (\alpha_l)_{ij}^{ep} dp_l - (\alpha_g)_{ij}^{ep} dp_g - (\alpha_{\varepsilon T_s})_{ij}^{ep} dT_s$$

(73a)

$$\frac{dm_l}{\rho_{lo}} = (\bar{\alpha}_l)_{kl}^{ep} d\varepsilon_{kl} + a_{ll}^{ep} dp_l - a_{gl}^{ep} dp_g - a_{lT_s}^{ep} dT_s - a_{lT_l} dT_l - a_{lT_g} dT_g$$

(73b)

$$\frac{dm_g}{\rho_{go}} = (\bar{\alpha}_g)_{kl}^{ep} d\varepsilon_{kl} - a_{gl}^{ep} dp_l + a_{gg}^{ep} dp_g - a_{gT_s}^{ep} dT_s - a_{gT_l} dT_l - a_{gT_g} dT_g$$

(73c)

$$d\eta_s = (\alpha_{T_s\varepsilon})_{kl}^{ep} d\varepsilon_{kl} - a_{T_s l}^{ep} dp_l - a_{T_s g}^{ep} dp_g + a_{T_s T_s}^{ep} dT_s$$

(73d)

$$d\eta_l = -a_{T_l l} dp_l - a_{T_l g} dp_g + a_{T_l T_l} dT_l$$

(73e)

$$d\eta_g = -a_{T_g l} dp_l - a_{T_g g} dp_g + a_{T_g T_g} dT_g$$

(73f)

with

$$D_{ijkl}^{\text{ep}} = D_{ijkl} - \frac{1}{H} D_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{pq}} D_{pqkl}$$

$$(\alpha_{\text{l}})_{ij}^{\text{ep}} = \alpha_{\text{l}} \delta_{ij} - \frac{1}{H} \left(D_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \right) \frac{\partial f}{\partial s}, \qquad (\bar{\alpha}_{\text{l}})_{kl}^{\text{ep}} = \alpha_{\text{l}} \delta_{kl} - \frac{1}{H} \left(D_{klmn} \frac{\partial f}{\partial \sigma_{mn}} \right) \frac{\partial g}{\partial s}$$

$$(\alpha_{\text{g}})_{ij}^{\text{ep}} = \alpha_{\text{g}} \delta_{ij} + \frac{1}{H} \left(D_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \right) \frac{\partial f}{\partial s}, \qquad (\bar{\alpha}_{\text{g}})_{kl}^{\text{ep}} = \alpha_{\text{g}} \delta_{kl} + \frac{1}{H} \left(D_{klmn} \frac{\partial f}{\partial \sigma_{mn}} \right) \frac{\partial g}{\partial s}$$

$$(\alpha_{\varepsilon T_{\text{s}}})_{ij}^{\text{ep}} = D_{ijkk}^{\text{ep}} \gamma + \frac{1}{H} D_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial f}{\partial T_{\text{s}}}, \qquad (\alpha_{T_{\text{s}}\varepsilon})_{kl}^{\text{ep}} = D_{klii}^{\text{ep}} \gamma + \frac{1}{H} D_{klmn} \frac{\partial f}{\partial \sigma_{mn}} \frac{\partial g}{\partial T_{\text{s}}}$$

$$a_{\text{ll}}^{\text{ep}} = a_{\text{ll}} + \frac{1}{H} \frac{\partial f}{\partial s} \frac{\partial g}{\partial s}, \qquad a_{\text{gg}}^{\text{ep}} = a_{\text{gg}} + \frac{1}{H} \frac{\partial f}{\partial s} \frac{\partial g}{\partial s} \qquad\qquad (74)$$

$$a_{T_{\text{s}}T_{\text{s}}}^{\text{ep}} = a_{T_{\text{s}}T_{\text{s}}} + \frac{1}{H} \left(\frac{\partial g}{\partial \sigma_{ij}} D_{ijkk} \gamma - \frac{\partial g}{\partial T_{\text{s}}} \right) \left(\frac{\partial f}{\partial \sigma_{ij}} D_{ijkk} \gamma - \frac{\partial f}{\partial T_{\text{s}}} \right)$$

$$a_{\text{lg}}^{\text{ep}} = a_{\text{gl}}^{\text{ep}} = a_{\text{lg}} - \frac{1}{H} \frac{\partial f}{\partial s} \frac{\partial g}{\partial s}$$

$$a_{\text{l}T_{\text{s}}}^{\text{ep}} = a_{\text{l}T_{\text{s}}} - \frac{1}{H} \left(\frac{\partial f}{\partial \sigma_{ij}} D_{ijkk} \gamma - \frac{\partial f}{\partial s} \right) \frac{\partial g}{\partial s}, \qquad a_{T_{\text{s}}\text{l}}^{\text{ep}} = a_{T_{\text{s}}\text{l}} - \frac{1}{H} \left(\frac{\partial g}{\partial \sigma_{ij}} D_{ijkk} \gamma - \frac{\partial g}{\partial s} \right) \frac{\partial f}{\partial s}$$

$$a_{\text{g}T_{\text{s}}}^{\text{ep}} = a_{\text{g}T_{\text{s}}} + \frac{1}{H} \left(\frac{\partial f}{\partial \sigma_{ij}} D_{ijkk} \gamma - \frac{\partial f}{\partial s} \right) \frac{\partial g}{\partial s}, \qquad a_{T_{\text{s}}\text{g}}^{\text{ep}} = a_{T_{\text{s}}\text{g}} + \frac{1}{H} \left(\frac{\partial g}{\partial \sigma_{ij}} D_{ijkk} \gamma - \frac{\partial g}{\partial s} \right) \frac{\partial f}{\partial s}$$

Imposing the restriction $\partial g / \partial s = 0$ in Eq. (66), we have,

$$(\bar{\alpha}_{\text{l}})_{ij}^{\text{ep}} = \alpha_{\text{l}} \delta_{ij}, \qquad (\bar{\alpha}_{\text{g}})_{ij}^{\text{ep}} = \alpha_{\text{g}} \delta_{ij}$$

$$a_{T_{\text{s}}\text{l}}^{\text{ep}} = a_{\text{l}T_{\text{s}}} - \frac{1}{H} \frac{\partial g}{\partial \sigma_{ij}} D_{ijkk} \gamma \frac{\partial f}{\partial s}, \qquad a_{T_{\text{s}}\text{g}}^{\text{ep}} = a_{T_{\text{s}}\text{g}} + \frac{1}{H} \frac{\partial g}{\partial \sigma_{ij}} D_{ijkk} \gamma \frac{\partial f}{\partial s} \qquad\qquad (75)$$

$$a_{\text{ll}}^{\text{ep}} = a_{\text{ll}}, \qquad a_{\text{gg}}^{\text{ep}} = a_{\text{gg}}, \qquad a_{\text{lg}}^{\text{ep}} = a_{\text{gl}}^{\text{ep}} = a_{\text{lg}} = a_{\text{gl}}, \qquad a_{\text{l}T_{\text{s}}}^{\text{ep}} = a_{\text{l}T_{\text{s}}}, \qquad a_{\text{g}T_{\text{s}}}^{\text{ep}} = a_{\text{g}T_{\text{s}}} = a_{T_{\text{s}}\text{g}}$$

In all cases, note that the evolution of the entropies in the fluid phases (73e), (73f) are unaffected by plasticity. With the assumption (67), the constitutive equations for the fluid mass contents retain their form for the elastic behaviour, although plasticity enters through the presence of the strain ε_{kl}. The only co-efficients modified by plasticity are those describing the evolution of total stress (73a) and entropy of solid (73d) through D_{ijkl}^{ep}, $(\alpha_{\text{l}})_{ij}^{\text{ep}}$, $(\alpha_{\text{g}})_{ij}^{\text{ep}}$, $(a_{\varepsilon T_{\text{s}}})_{ij}^{\text{ep}}$, $(a_{T_{\text{s}}\varepsilon})_{kl}^{\text{ep}}$, $a_{T_{\text{s}}\text{l}}^{\text{ep}}$, $a_{T_{\text{s}}\text{g}}^{\text{ep}}$, and $a_{T_{\text{s}}T_{\text{s}}}^{\text{ep}}$.

Now, substituting the elastic constitutive coefficients in Eqs. (44a)–(44g) by their counterparts in Eqs. (73a)–(75) yields the differential equations governing the elasto-plastic thermo-hydro-mechanical response in unsaturated porous media.

9.3. Modified Cam-Clay as a plastic driver

To further define the basic features of the present plasticity model, the modified Cam-Clay with isotropic hardening is invoked as a plasticity driver. Matric suction, s, solid temperature, T_{s}, and plastic volumetric strain, $\varepsilon_{ii}^{\text{pl}}$, are introduced as hardening variables. The yield surface is defined as,

$$f(p', q, P_{\text{c}}) = \frac{q^2}{M^2 p'} + p' - P_{\text{c}} = 0 \qquad\qquad (76)$$

in which $p' = -\sigma_{ii}'/3$ is the mean effective stress, q is the second invariant of stress deviator, and M is the slope of the critical state line, which is related to the internal friction angle ϕ'; M is assumed to be independent of temperature and suction. P_{c} is the isotropic yield stress or the pre-consolidation pressure, fixing
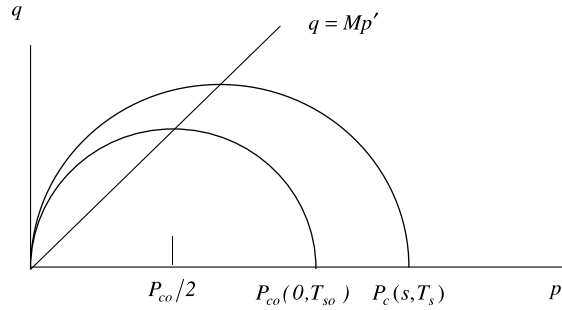
Fig. 2. Plasticity model.

the size of the yield surface (Fig. 2). It incorporates the effect of strain hardening and strain softening due to plastic volume change through the hardening parameter $X = -\varepsilon_{ii}^{\mathrm{pl}}$. Hardening occurs due to plastic contractancy $\mathrm{d}X > 0$, and softening occurs due to plastic dilatancy $\mathrm{d}X < 0$. P_{c} also incorporates the effect of the stiffening and softening of the solid matrix with suction and temperature. Much work has been conducted on the stiffening effect of suction on the soil matrix; however, the work on the effect of temperature on the soil matrix has been relatively limited. This is particularly the case on the combined effect of suction and temperature on the pre-consolidation stress P_{c}. There is currently no experimental evidence as to the coupled effect of suction and temperature on the soil response. The hydro-thermo-mechanical hardening law for unsaturated soils may be defined as,

$$P_{\mathrm{c}}(s, T_{\mathrm{s}}) = P_{\mathrm{co}} \Psi(s, T_{\mathrm{s}}) \exp\left(\frac{v_{\mathrm{o}} X}{\lambda(s, T_{\mathrm{s}}) - \kappa}\right) \tag{77}$$

in which $v_{\mathrm{o}} = 1 + e_{\mathrm{o}}$ is the initial specific volume, $\lambda(s, T_{\mathrm{s}})$ is the current slope of $v \sim \ln p'$ curve at loading, and κ is the current slope of $v \sim \ln p'$ curve at unloading, assumed to be independent of temperature and suction. P_{co} is the saturated isotropic yield stress at the reference solid temperature T_{so}, and $\Psi(s, T_{\mathrm{s}})$ is the hydro-thermal hardening multiplier, describing the evolution of yield surface with suction and solid temperature (Appendix A),

$$\Psi(s, T_{\mathrm{s}}) = \exp\left[\frac{N(s, T_{\mathrm{s}}) - N(0, T_{\mathrm{so}})}{\lambda(s, T_{\mathrm{s}}) - \kappa} - \frac{\lambda(s, T_{\mathrm{s}}) - \lambda(0, T_{\mathrm{so}})}{\lambda(s, T_{\mathrm{s}}) - \kappa} \ln P_{\mathrm{co}} - \frac{1 + e_{\mathrm{o}}}{\lambda(s, T_{\mathrm{s}}) - \kappa} c_{\mathrm{T}} \Delta T_{\mathrm{s}}\right] \tag{78}$$

where $\lambda(0, T_{\mathrm{so}})$ is the slope of $v \sim \ln p'$ curve at saturation, and N is the specific volume at $p' = 1$ unit. According to Eq. (78), suction and temperature may affect the size of the yield surface through both the temperature and suction dependency of λ and N, as well as the thermal expansion coefficient of the solid skeleton. The change in the size of the yield surface due to thermal dependency of λ is often referred to as thermal ductilisation/softening (Hueckel et al., 1998). For $\lambda(s, T_{\mathrm{s}}) = \lambda(0, T_{\mathrm{so}})$, there will be no thermal ductilisation, and the change in the yield stress will result in parallel consolidation lines in a logarithmic scale (Fig. 3). For $\lambda(s, T_{\mathrm{s}}) > \lambda(0, T_{\mathrm{so}})$ the consolidation lines will be convergent. The isothermal, suction-controlled, experimental results of Cui and Delage (1996) correspond to the former response while those of Wheeler and Sivakumar (1995) correspond to the latter condition. The functions $N(s, T_{\mathrm{s}})$ and $\lambda(s, T_{\mathrm{s}})$, controlling the evolution of the yield surface with temperature and suction, are determined experimentally. To account for the isothermal elastic response of the soil skeleton that occurs with increasing suction after the air entry value (Fleureau et al., 1993), as well as the collapse phenomenon upon wetting (Matyas and Radhakrishna, 1968), the slope of the mean effective stress $\partial p'/\partial s$ must be smaller than the slope of the pre-consolidation stress $\partial P_{\mathrm{c}}/\partial s$ (Loret and Khalili, 2000a),

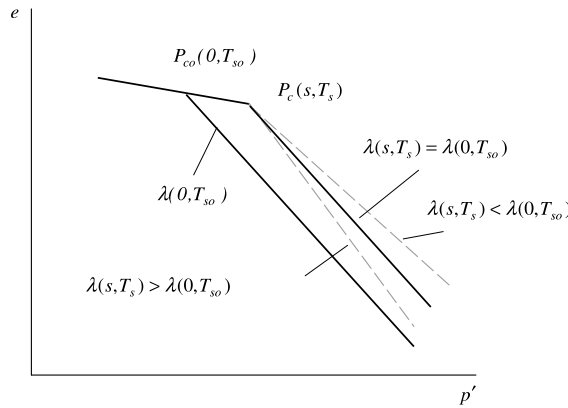Fig. 3. Isothermal constant-suction consolidation lines for unsaturated porous media.

$$\left.\frac{\partial p'}{\partial s}\right|_{T_s} < \left.\frac{\partial P_c}{\partial s}\right|_{T_s} \tag{79}$$

Now using Eq. (77), the hardening modulus, h, in Eq. (72) becomes,

$$h = \frac{v_o}{\lambda(s, T_s) - \kappa} P_{co} \Psi(s, T_s) \exp\left(\frac{v_o X}{\lambda(s, T_s) - \kappa}\right) \frac{\partial g}{\partial p'} \tag{80}$$

As discussed in Section 9.1, the flow rule is non-associative with respect to suction. Assuming associativity in the effective stress space, the plastic potential is defined as,

$$g(p', q, \vartheta) = \frac{q^2}{M^2 p'} + p' - \vartheta = 0 \tag{81}$$

in which ϑ is a function of solid temperature, to be determined experimentally. For the particular case of associativity with respect to the solid temperature, ϑ may be determined using the constraints,

$$\frac{\partial \vartheta(T_s)}{\partial T_s} = \frac{\partial P_c(s, T_s)}{\partial T_s} \quad \text{and} \quad \frac{\partial \vartheta(T_s)}{\partial s} = 0 \tag{82}$$

Acknowledgements

Appendix A. Hydro-thermal hardening function

To derive the hydro-thermal hardening function, consider consolidation responses at constant suction and constant solid temperature of two identical soil samples, and the stress path 1–2–3, as shown on Fig. 4. Sample A is consolidated in a saturated state and at the reference solid temperature T_{so}, whereas Sample B is consolidated at the solid temperature T_s, suction s and the average liquid–gas interface temperature, T, with $s > s_e(T)$. Consider the stress path 1–2–3–4–5–6. The stress at Point 1 is P_{co}, and at Point 6 it is
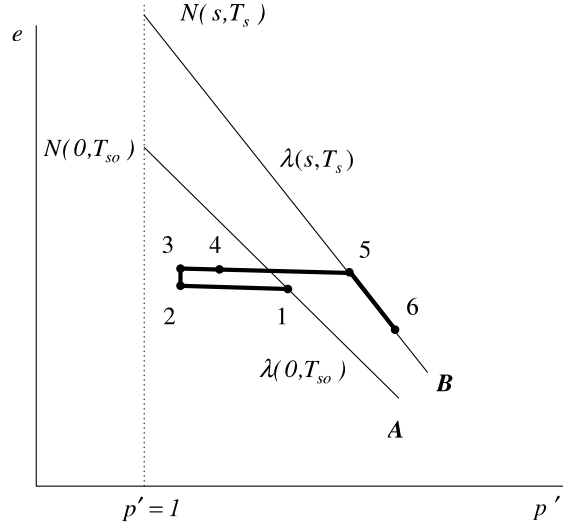
Fig. 4. Hydro-thermal hardening in unsaturated media.

$P_c(s, T_s)$. At Point 1, the sample is isothermally unloaded to Point 2, and then loaded thermally (i.e. at constant effective stress) to solid temperature T_s at Point 3. Along the stress path 3–4 the suction within the sample is increased to s. From 4 to 6 the sample is loaded mechanically to effective stresses p'_5 and $P_c(s, T_s)$, at Points 5 and 6, respectively.

The plastic volumetric change from 5 to 6 can be written as:

$$\varepsilon_{ii}^{pl} = \varepsilon_{ii} - \varepsilon_{ii}^{el} = \frac{1}{1 + e_o}[\lambda(s, T_s) - \kappa] \ln \frac{P_c(s, T_s)}{p'_5} \tag{A.1}$$

The void ratio, e_5, at Point 5 is expressed as,

$$e_5 = N(s, T_s) - \lambda(s, T_s) \ln p'_5 = N(0, T_{so}) - \lambda(0, T_{so}) \ln P_{co} - \kappa \ln \frac{p'_5}{P_{co}} + c_T \Delta T_s \tag{A.2}$$

in which,

$$\Delta T_s = T_s - T_{so} \tag{A.3}$$

Substituting for p'_5 in Eq. (A.1) from Eq. (A.2) and rearranging give,

$$P_c(s, T_s) = \exp \left[\frac{N(s, T_s) - N(0, T_{so})}{\lambda(s, T_s) - \kappa} + \frac{\lambda(0, T_{so}) - \kappa}{\lambda(s, T_s) - \kappa} \ln P_{co} - \frac{(1 + e_o)(\varepsilon_{ii}^{pl} + c_T \Delta T_s)}{\lambda(s, T_s) - \kappa} \right] \tag{A.4}$$

Eq. (A.4) can be further expressed as,

$$P_c(s, T_s) = \exp \left[\frac{N(s, T_s) - N(0, T_{so})}{\lambda(s, T_s) - \kappa} - \frac{\lambda(s, T_s) - \lambda(0, T_{so})}{\lambda(s, T_s) - \kappa} \ln P_{co} + \ln P_{co} - \frac{(1 + e_o)(\varepsilon_{ii}^{pl} + c_T \Delta T_s)}{\lambda(s, T_s) - \kappa} \right] \tag{A.5}$$

or

$$P_{\mathrm{c}}(s, T_{\mathrm{s}}) = P_{\mathrm{co}} \exp\left[-\frac{(1+e_{\mathrm{o}})\varepsilon_{ii}^{\mathrm{pl}}}{\lambda(s, T_{\mathrm{s}}) - \kappa} \right] \times \Psi(s, T_{\mathrm{s}}) \tag{A.6}$$

in which

$$\Psi(s, T_{\mathrm{s}}) = \exp\left[\frac{N(s, T_{\mathrm{s}}) - N(0, T_{\mathrm{so}})}{\lambda(s, T_{\mathrm{s}}) - \kappa} - \frac{\lambda(s, T_{\mathrm{s}}) - \lambda(0, T_{\mathrm{so}})}{\lambda(s, T_{\mathrm{s}}) - \kappa} \ln P_{\mathrm{co}} - \frac{(1+e_{\mathrm{o}})c_{\mathrm{T}}\Delta T_{\mathrm{s}}}{\lambda(s, T_{\mathrm{s}}) - \kappa} \right] \tag{A.7}$$

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